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A SURVEY OF FABER METHODS IN NUMERICAL APPROXIMATION

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Abstract—Although well known in function theory, Faber polynomials and the Faber transform have only recently been systematically applied for numerical polynomial and rational approximation of complex valued functions. This paper surveys some recent work of this author and others in this area.

INTRODUCTION

The purpose of this paper is to survey the use of Faber methods in numerical approximation in the complex plane. In this section we provide a brief introduction to Faber polynomials and the Faber transform. Readers requiring more information on the theory should consult one of the standard texts and surveys such as [5], [7, Chap. 3, Vol. 3] or [8, Chap. 1, Section 6]. Additional material including computer drawn plots of the boundary behaviour of Faber polynomials is given in [10], and some of the deeper theoretical results known may be found in the referenced papers of Pommerenke, Kövari and Anderson [1, 2, 6, 9]. In Section 2 we discuss some numerical techniques for handling Faber series, this section being essentially a summary of the paper [11]. Finally in Section 3 we discuss the CF method for polynomial approximation, and Padé and CF methods for rational approximation.

For simplicity we let D denote a Jordan domain bounded by a rectifiable Jordan curve C. (However it is possible to consider much more general sets: see for example [9].) By the Riemann mapping theorem there exists a conformal map ϕ of the complement of $D \cup C$ onto $|w| > 1$ with the property that

$$\lim_{z \to \infty} \frac{\phi(z)}{z} \quad \text{is real and positive.}$$

For F analytic on $|w| < 1$ continuous on the closure, the Faber transform of F is defined by

$$f(z) = T(F)(z) = \frac{1}{2\pi i} \int_C \frac{F(\phi(\xi))}{\xi - z} \, \mathrm{d}\xi \tag{1.1}$$

for z in D. Denoting the inverse of ϕ by ψ, it is therefore easily seen that

1.1 Lemma

Let F be analytic on the closed disc $|w| \leqslant 1$, and let $f = T(F)$. Then $F(w) - f[\psi(w)]$ can be extended analytically to $|w| > 1$ including the point at ∞. ■

We call this the "singularity preserving property". In particular therefore we observe that $\phi_n(z) = T(w^n)$ is a polynomial of degree n: the Faber polynomial.

If

$$F(w) = \sum_{n=0}^{\infty} a_n w^n$$

then (at least formally), $f = T(F)$ has the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n \phi_n(z) \tag{1.2}$$

indeed this property is often taken as the definition of the Faber transform. This expansion of f is known as the Faber series, and it forms the subject of Section 2.

2. POLYNOMIAL APPROXIMATION BY TRUNCATING THE FABER SERIES

2.1. *The Faber projection*

The Faber series (1.2) can be shown [7] to be uniformly and absolutely convergent on the closed region $D \cup C$ provided f is analytic on this region: convergence under weaker conditions is considered in [2].

More importantly for computational purposes, the mapping $P_n(f)$ defined by

$$P_n(f)(z) = \sum_{j=0}^{n} a_j \phi_j(z)$$

defines a projection of the space of functions F analytic on D and continuous on the closure, equipped with the uniform norm, onto the subspace of polynomials of degree n. We introduce a constant V known as the total rotation of the Jordan curve C. This is defined (provided it is finite) to be the total variation of the angle of the tangent to C as the curve is described anticlockwise once. For instance, if D is convex, this is 2π. Then, (even if the Faber series does not converge) we have

$$\| P_n \| < \frac{V}{\pi}\left\{\frac{4}{\pi^2} \ln n + 1.773\right\}, \quad n > 0. \tag{2.1}$$

(See [2, 11]. Using a now standard argument [3, 4] we deduce that if p is any polynomial of degree n

$$\|f - P_n(f)\|_\infty = \|f - p + P_n(p - f)\|_\infty \leqslant (1 + \|P_n\|)\|f - p\|_\infty \tag{2.2}$$

where the norm is of course the uniform norm. Since here p could be the best approximation in the uniform sense to f, it follows that the error of the truncated Faber series is within a ratio $1 + (V/\pi)[(4/\pi^2) \ln n + 1.773]$ of that of the best approximation. Although this ratio $\to \infty$ with n, for small values of n it is not too large and hence the truncated Faber series provides a good approximation to f. Moreover numerical experiments [11, 13] indicate that if f is well behaved (and if it is not, there is little point in practice in computing a polynomial approximation) then this bound on the ratio is often very conservative.

2.2. *Numerical computations with the Faber series*

Provided the conformal map ϕ (or its inverse ψ) is known, the Faber polynomials are easy to compute numerically. If

$$\phi_n(z) = \sum_{j=0}^{n} c_j z^j,$$

and recalling that ϕ_n is the Faber transform of w^n, we have from (1.1) together with the Cauchy integral formula

$$c_j = \frac{1}{2\pi i} \int_{|z|=s} \frac{(\phi(z))^n}{z^{j+1}} \,\mathrm{d}z \tag{2.3}$$

where s is chosen sufficiently large that C is completely enclosed within the circle of integration. (In practice,

$$s = 1.1 \max|z|, \quad z \in C$$

is usually suitable.) Replacing the integral by its N-point trapezium rule approximation yields

$$c_j \approx \frac{1}{Ns^j} \sum_{k=0}^{N-1} [\phi(s\mathrm{e}^{2\pi ik/N})]^n \,\mathrm{e}^{-2\pi ijk/N}.$$

Thus the coefficients can be efficiently and accurately computed by means of the fast Fourier transform.

An alternative way to compute the Faber polynomials is recursively from the expansion of the inverse map ψ at ∞. If

$$\psi(w) = b_{-1}w + b_0 + b_1 w^{-1} + b_2 w^{-2} + \cdots$$

then the Faber polynomials satisfy the well known recursion

$$b_{-1}\phi_{n+1}(z) = z\phi_n(z) - \sum_{j=0}^{n} b_j \phi_{n-j}(z) - nb_n, \quad n \geqslant 0, \quad \phi_0 = 1. \tag{2.4}$$

(See e.g. [11].) This formula is particularly useful for polygonal regions for which the expansion of ψ may be obtained by means of the Schwarz–Christoffel transformation.

2.3. Computing the Faber coefficients of a specified function

Having computed the Faber polynomials for the region D, it is only necessary to compute the actual expansion coefficients a_j to obtain our polynomial approximation to a given function f. It is well known (see e.g. [7, 8]) that

$$a_j = \frac{1}{2\pi i}\int_{|w|=1} \frac{f(\psi(w))}{w^{j+1}}\,\mathrm{d}w. \tag{2.5}$$

Provided f is analytic on the closure of D, it is convenient to avoid boundary singularities in ψ by taking the integral around $|w| = s$, where $s > 1$ but small enough to not to encompass any singularities of f. A value of s of about 1.1 is usually suitable in practice. The coefficients are now evaluated by applying the trapezium rule and using the fast Fourier transform, as described above for the coefficients of the Faber polynomials, and by truncating the series at an appropriate point one obtains a near best polynomial approximation to f.

The methods described in this section are given in considerably more detail, with the results of numerical experiments, in [11]. (Note however that in that paper a different normalisation, following [7] is used for the Faber polynomials. The normalisation used in this author's subsequent papers, including this one, was adopted as being the more common current practice.)

3. PADÉ AND CF METHODS

3.1. The Faber–Padé approximant

The extension of Faber methods to rational approximation is made possible by the singularity preserving property of the Faber transform.

3.1. Proposition

Let R be a type (m, n) rational function with all its poles in $|w| > 1$. Then $r = T(R)$ is a type $(\tilde{m}, n)$ rational function, where $\tilde{m} = \max(m, n-1)$ and moreover its poles are the images under ψ of those of R.

Proof: It follows from (1.1) and Lemma 1.1 that r is meromorphic in the extended plane and hence rational. The second part of the theorem is a trivial consequence of Lemma 1.1. ■

Observe that if R is a normal type(m, n) Padé approximant to F, and has all its poles in $|w| > 1$, then $r = T(R)$ satisfies

$$f(z) - r(z) = \sum_{k=m+n+1}^{\infty} d_k \phi_k(z)$$

for some coefficients d_k. This is the Faber–Padé approximant introduced in [12]. There are two problems with this approximant: firstly the fact that it is not of the "correct" type if $m < n - 1$ and secondly the condition that R have no poles in the unit disc. Although as is well known the first problem can be overcome in the special case of Chebyshev–Padé approximation, there does not seem to be any easy way to extend this to the Faber case, and we will not consider this further. However the following Montessus type theorem guarantees the existence of the Faber–Padé approximant for meromorphic f under certain conditions and also shows that it behaves in the expected manner. The proof is a straightforward application of the singularity preserving property (Lemma 1.1) and it is possible to generalise other properties of the classical Padé approximants in a similar fashion. The theorem first appeared in [14], which contains details of the proof.

3.2. Theorem

Let f be analytic in the closure of D and meromorphic in the Jordan region D_s bounded by the level curve $\phi(z) = s$, $s > 1$, with precisely m poles (counted according to multiplicities). Then for n sufficiently large the type(m, n) Faber–Padé approximant r_{mn} exists, and as $n \to \infty$, r_{mn} converges uniformly to f on any compact subset of D which excludes the poles of f.

3.2. Calculating the transform of a rational function

A crucial stage in computing Faber–Padé or Faber–CF approximants is the computation of the transform of a rational function R analytic on the disc. In [12] this was carried out by computing the poles of R and applying Proposition 3.1 in an obvious fashion. We describe here an alternative and much faster method, first given in [14] based on the integral representation (1.1) of the transform. In fact for any $f = T(F)$ we have, assuming without loss of generality that $0 \in D$,

$$\frac{f^{[k]}(0)}{k!} = \frac{-1}{4\pi^2} \int_{|z|=t} \frac{1}{z^{k+1}} \int_C \frac{F(\phi(\xi))}{\xi - z}\, d\xi\, dz$$

where $t > 0$ is small. By interchanging the order and performing the z integration we obtain

$$\frac{f^{[k]}(0)}{k!} = \frac{1}{2\pi i} \int_C \frac{F(\phi(\xi))}{\xi^{k+1}}\, d\xi.$$

If F is entire we may replace C by a circle and evaluate as many of the integrals as we require simultaneously by the trapezium rule and the fast Fourier transform. However this is not in general possible for the case required here where $F = R$ is a type (m, n) rational function. Instead we make the substitution $\xi = \psi(w)$ yielding, for $r = T(R)$

$$\frac{r^{[k]}(0)}{k!} = \frac{1}{2\pi i} \int_{|w|=s} \frac{R(w)\psi_{,}(w)}{(\psi(w))^{k+1}}\, dw. \tag{3.1}$$

Here $s > 1$ is chosen sufficiently small to ensure that the circle $|w| = s$ does not enclose any poles of R. To calculate the transform of a type (m, n) rational function we evaluate the integrals (3.1) for $k = 0, 1, \ldots m + n + 1$, by the trapezium rule. Details of numerical experiments, which indicate that the technique is both efficient and extremely accurate, are given in [14].

3.3. Faber–CF approximations

The CF method is an extremely effective way of approximating analytic functions by polynomials or rational functions. The initials CF stand for Carathéodory–Fejér, on whose theorem the basic polynomial version of this method is based. Although the method is related to some techniques for approximation on the real line, it is now appreciated that its fundamental form is on the unit disc. An approximation is computed by applying the singular value decomposition to a Hankel matrix formed from the Maclaurin coefficients of the function to be approximated. Since the method itself has nothing to do with Faber techniques, we will not give the details here, but note that both on the disc and indeed on the real line the method usually produces an approximation within a small fraction of a percent of the best: in fact generally in practice better than any theoretically "best" approximation obtained by a descent algorithm! A good introduction to the method (on the disc and the real line) is given in [15]. Using the Faber transform, extension to our general region D bounded by C is straightforward. First compute the Faber coefficients of the function f to be approximated, as described in Section 2.3. (The Faber polynomials are not needed.) These are precisely the Maclaurin coefficients of F, where $F = T(f)$: see (1.2). Then compute a polynomial or rational CF approximation to F, and compute the Faber transform of this approximation, as in Section 3.2.

The polynomial Faber–CF method was analysed in some detail, and tested numerically, in [13]. The major conclusions were as follows:

(i) If C is an analytic curve, the CF method is almost as effective as for the disc, and generally produces virtually best approximations.
(ii) If D is convex, the method generally produces an approximation within a ratio α of best, where $\alpha\pi$ is the largest external angle at a corner of C. (Note that necessarily, $1 \leqslant \alpha < 2$.)

Unfortunately, in case (ii), this means that in practice the approximation is often worse than that obtained simply by truncating the Faber series. However, the techniques is much more useful for rational approximation. Rational Faber–CF approximation was introduced in [12], in the same paper as the Faber–Padé approximants. Although not analysed in such detail as for the polynomial case, (since analysis of the rational CF method even on the disc is quite delicate) it was found experimentally that case (ii) also holds for the rational method. (No experiments were carried out for the rational method for analytic boundary curves, since these seem to have little practical application.) The approximations obtained gave in most cases very much smaller errors than the corresponding Faber–Padé approximants, while requiring much less computing time than computing a true best approximation by a descent algorithm. (Of course if a best approximation is required, the Faber–CF approximation is a very good starting point for the descent.)

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